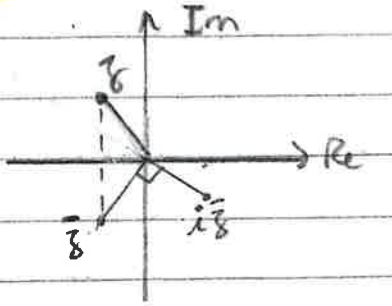
SGS Mathematics Extension 2 Trial Solutions 2024

Section I M/C

①



D

② $\text{Proj}_{\underset{\sim}{v}} \underset{\sim}{u} = \dfrac{1\times 0 + 2\times 1 + 0\times 3}{1^2+3^2}\left(\underset{\sim}{j} + 3\underset{\sim}{k}\right)$

$= \frac{1}{5}\left(\underset{\sim}{j} + 3\underset{\sim}{k}\right)$ **B**

③ $v^2 = n^2\left(a^2 - (x-c)^2\right)$

$4^2 = n^2\left(3^2 - (1)^2\right)$

$n^2 = 2$

$n = \sqrt{2} \qquad T = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$ **C**

④ $8 \equiv A(x-1)^2 + B(x+1)(x-1) + C(x+1)$

$x = 1 \Rightarrow 8 = 2C \Rightarrow C = 4$

$x = -1 \Rightarrow 8 = 4A \Rightarrow A = 2$ **A**

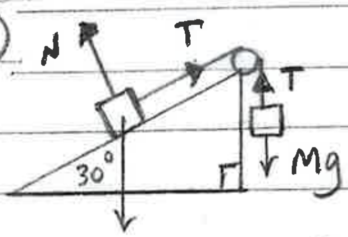
Coeff of $x^2 \Rightarrow A + B = 0 \Rightarrow B = -2$

⑤ $\alpha = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^\circ$, $\beta = \cos^{-1}\left(-\frac{1}{2}\right) = 120^\circ$

$\gamma = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$, $\alpha + \beta + \gamma = 225^\circ$

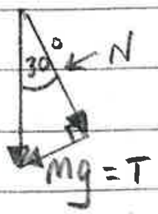
D

⑥



Equilibrium $\Rightarrow$

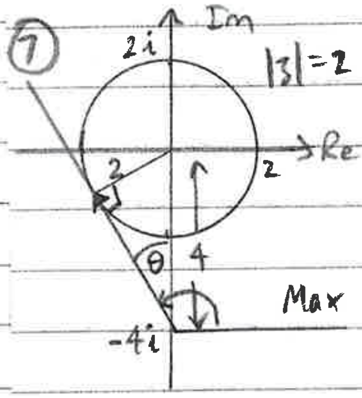
$T = Mg$



$\frac{N}{Mg} = \cot 30^\circ$

$N = \sqrt{3}Mg$ **C**

⑦



$|z| = 2$

$\sin\theta = \frac{2}{4} = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

Max $\text{Arg}(z+4i) = \frac{\pi}{2} + \frac{\pi}{6}$

$= \frac{2\pi}{3}$

C

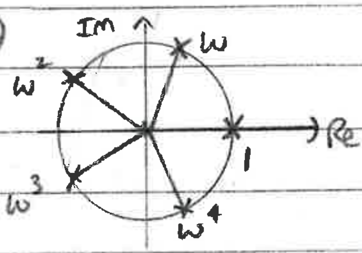
⑧ $p \Rightarrow q$ is false (3 uppermost)

Negation: $p \wedge \sim q$ True $\sim(p \Rightarrow q)$

Converse: $q \Rightarrow p$ True as 6 is divis by 3.

A

⑨



$\text{Im}(w + w^4) = 0$

D

⑩ $\forall y, \exists x : x^2 - y^2 = x$

$x^2 - x = y^2$

$\left(x - \frac{1}{2}\right)^2 = y^2 + \frac{1}{4}$

RHS $\geq \frac{1}{4}$ $\forall y$, so always solutions for x.

Not the case for the others **A**

Overall **DBCADCCADA**

Section II

(11)

(a) $z = a + 2i$, $w = 1 - ai$

(i) $zw = (a+2i)(1-ai)$

$= a - a^2 i + 2i + 2a$

$= 3a + (2-a^2)i$ ✓

(ii) $z - aw = a + 2i - a(1-ai)$

$= a + 2i - a + a^2 i$

$= (a^2+2)i$

$= (a^2+2)\text{cis}\frac{\pi}{2}$ ✓

(iii) $(\bar{w})^2 + 2w = (1+ai)^2 + 2(1-ai)$

$= 1 + 2ai - a^2 + 2 - 2ai$

$= 3 - a^2$ ✓ (Real)

(b)

(i) $\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{e^x}{1+(e^x)^2}\,dx$

$= \tan^{-1}(e^x) + c$ ✓

(ii)

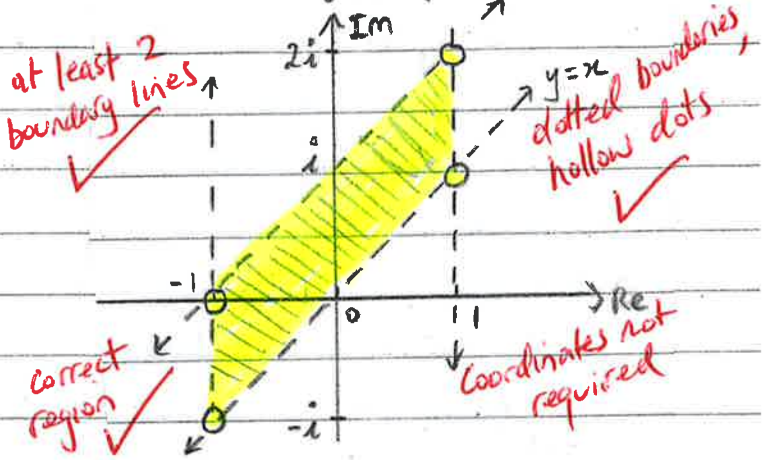
$\int \sin^4 x \sin 2x\,dx = \int \sin^4 x\,(2\sin x \cos x)\,dx$

$= 2\int \sin^5 x \cos x\,dx$

$= \frac{1}{3}\sin^6 x + c$ ✓

(c) $Re(z) \le 1$, $Re(z) < Im(z)$

$-\frac{\pi}{2} < Arg(z+1) < \frac{\pi}{4}$



(d) $t=0$, $x=0$, $v=2$

(i) $v = x^2 + 1$, $a = v \cdot \frac{dv}{dx}$

$= (x^2+1)(2x)$ ✓

$x = 3 \Rightarrow a = (3^2+1)(2\times 3)$

$= 60$ ✓

(ii) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = x^2 + 1$

$\frac{1}{2}v^2 = \frac{1}{3}x^3 + x + c$ ✓

$t=0$, $x=0$, $v=2 \Rightarrow c = 2$

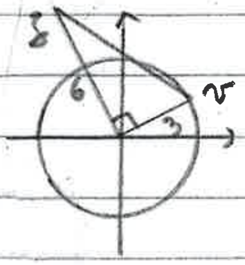
$v^2 = \frac{2}{3}x^3 + 2x + 4$ ✓

$a \ge 1$ so $v \ge 2$ $\Rightarrow$ $v = \sqrt{\frac{2}{3}x^3 + 2x + 4}$ ✓ (positive only)

$x = 3 \Rightarrow v = \sqrt{\frac{2}{3}(3)^3 + 2(3) + 4} = \sqrt{28}$

$= 2\sqrt{7}$ ✓

(e)



$u = 2i$, $|v| = 3$, $z = uv$

$|z - v| = \sqrt{3^2 + 6^2}$ ✓

$= \sqrt{45}$ or $3\sqrt{5}$ ✓

OR

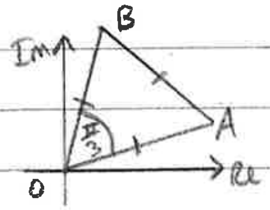
$|z - v| = |v||u - 1|$

$= 3|-1+2i|$ ✓

$= 3\sqrt{5}$ ✓

(12)

(a)



$\vec{OB} = \text{cis}\frac{\pi}{3}\,\vec{OA}$

$z = \vec{OB} = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)(6+2i)$ ✓

$= 3 + i + 3\sqrt{3}i - \sqrt{3}$

$= (3-\sqrt{3}) + (1+3\sqrt{3})i$ ✓

(b) (i) $t = \tan\frac{x}{2} \Rightarrow x = \tan^{-1}(2t)$; $dx = \frac{2dt}{1+t^2}$

$x=0, t=0$

$x=\frac{\pi}{2}, t=1$

$I = \int_0^1 \frac{2dt}{1+t^2+1-t^2+2t}$ ✓

$= \int_0^1 \frac{dt}{1+t} = \left[\ln|1+t|\right]_0^1 = \ln 2$ ✓

(ii) $x = \frac{\pi}{2} - u \Rightarrow dx = -du$, $x=0, u=\frac{\pi}{2}$

$x=\frac{\pi}{2}, u=0$

$I = \int_{\frac{\pi}{2}}^{0} \frac{(\frac{\pi}{2}-u)(-du)}{1+\cos(\frac{\pi}{2}-u)+\sin(\frac{\pi}{2}-u)}$ ✓

$= \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}-u}{1+\sin u+\cos u}\,du$

$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{du}{1+\cos u+\sin u} - I$ ✓

$\therefore 2I = \frac{\pi}{2}\ln 2$ (from (i))

$I = \frac{\pi}{4}\ln 2$ ✓

(c) $z^4 - 2z^3 + 9z^2 - 6z + 18 = 0$, Root: $1+\sqrt{5}i$

Since the coefficients are real, complex roots come in conjugate pairs, hence $1-\sqrt{5}i$ is also a root. Let the others be α, β. ✓

Using sum and product of roots

$\alpha+\beta+1+i\sqrt{5}+1-i\sqrt{5} = 2$, $\alpha\beta(1+i\sqrt{5})(1-i\sqrt{5}) = 18$

$\alpha+\beta = 2$ $\quad\quad \alpha\beta(1+5) = 18$

$\alpha = -\beta$ (1) $\quad\quad \alpha\beta = 3$ (2) ✓ (both)

From (1) and (2) $\alpha^2 = -3 \Rightarrow \alpha = \pm\sqrt{3}i, \beta = \mp\sqrt{3}i$

Roots, $1\pm\sqrt{5}i$, $\pm\sqrt{3}i$ ✓

(d) $l: \underset{\sim}{r} = \begin{pmatrix}7\\4\\13\end{pmatrix} + \lambda\begin{pmatrix}2\\7\\10\end{pmatrix}$, $S: \left|\underset{\sim}{r} - \begin{pmatrix}3\\-1\\2\end{pmatrix}\right| = 3$

Sub l into S

$(7+2\lambda-3)^2 + (4+7\lambda+1)^2 + (13+10\lambda-2)^2 = 9$ ✓

$(4+2\lambda)^2 + (5+7\lambda)^2 + (11+10\lambda)^2 = 9$

$16+16\lambda+4\lambda^2+25+70\lambda+49\lambda^2+121+220\lambda+100\lambda^2 = 9$

$153\lambda^2 + 306\lambda + 153 = 0$

$\lambda^2 + 2\lambda + 1 = 0$

$(\lambda+1)^2 = 0$, $\lambda = -1$ ✓

One solution, touches at Q: $(7+2(-1), 4+7(-1), 13+10(-1))$

so Q is $(5,-3,3)$ ✓

(e) RTP $\forall x,y \in \mathbb{Z}$, if $x^2(y+3)$ is even, then x is even or y is odd.

Contrapositive: RTP If x is odd and y is even then $x^2(y+3)$ is not even (ie odd) ✓

Let $x = 2k+1, y = 2l$, $k, l \in \mathbb{Z}$

then $x^2(y+3) = (2k+1)^2(2l+3)$

$= (4k^2+4k+1)(2l+3)$

$= 8k^2l + 12k^2 + 4kl + 12k + 2l + 3$ ✓

$= 2(4k^2l + 6k^2 + 2kl + 6k + l + 1) + 1$

which is odd.

Hence proven

13 (a) $I=\int x^3(\log x)^2\,dx$ Let $u=(\log x)^2$, $v'=x^3$

$u'=2\log x\left(\frac{1}{x}\right)$, $v=\frac{1}{4}x^4$ ✓

$I=\frac{1}{4}x^4(\log x)^2-\frac{1}{2}\int x^3\log x\,dx$ ✓

Let $U=\log x$, $V'=x^3$

$U'=\frac{1}{x}$, $V=\frac{1}{4}x^4$

$I=\frac{1}{4}x^4(\log x)^2-\frac{1}{2}\left(\frac{1}{4}x^4\log x-\frac{1}{4}\int x^3\,dx\right)$ ✓

$=\frac{1}{4}x^4(\log x)^2-\frac{1}{8}x^4\log x+\frac{1}{32}x^4+c$

$=\frac{1}{32}x^4\left[8(\log x)^2-4\log x+1\right]+c$ ✓

(b) RTP $(\cos\theta+i\sin\theta)^n(\sin\theta+i\cos\theta)^n=e^{\frac{n i\pi}{2}}$

LHS $=\left[(\cos\theta+i\sin\theta)(\sin\theta+i\cos\theta)\right]^n$

$=\left[\cos\theta\sin\theta+i\cos^2\theta+i\sin^2\theta-\sin\theta\cos\theta\right]^n$ ✓

$=(i)^n$

$=\left(e^{i\frac{\pi}{2}}\right)^n$

$=e^{\frac{n i\pi}{2}}$ = RHS ✓

(c) (i) RTP $\frac{2}{(x+1)(x^2+1)}=\frac{1}{x+1}-\frac{x-1}{x^2+1}$

RHS $=\frac{x^2+1-(x-1)(x+1)}{(x+1)(x^2+1)}$

$=\frac{x^2+1-x^2+1}{(x+1)(x^2+1)}$ Show

$=\frac{2}{(x+1)(x^2+1)}$ = LHS ✓

(ii) $I_n=\int_0^1\frac{2x^n}{(x+1)(x^2+1)}\,dx$

(α) $I_0=\int_0^1\frac{2}{(x+1)(x^2+1)}\,dx$

$I_0=\int_0^1\left(\frac{1}{x+1}-\frac{x}{x^2+1}+\frac{1}{x^2+1}\right)dx$ (from (i))

$=\left[\ln|x+1|-\frac{1}{2}\ln|x^2+1|+\tan^{-1}x\right]_0^1$ ✓

$=\ln 2-\frac{1}{2}\ln 2+\frac{\pi}{4}-\left(0-\frac{1}{2}(0)+0\right)$

$=\frac{1}{2}\ln 2+\frac{\pi}{4}$. ✓

(β) $I_0+I_2=\int_0^1\frac{2(1+x^2)}{(x+1)(x^2+1)}\,dx$

$=\left[2\ln|x+1|\right]_0^1$

$=2\ln 2-0$

$=2\ln 2$ ✓

$\therefore I_2=2\ln 2-\left(\frac{1}{2}\ln 2+\frac{\pi}{4}\right)$

$=\frac{3}{2}\ln 2-\frac{\pi}{4}$. ✓

(d) (i) $(2m+3)^2=n^2+p$ true for

$m=2,\ n=6,\ p=13$ or equivalent ✓

(ii) Assume $\exists\, m\in\mathbb{Z}^+ : (5m+3)^2=n^2+p$,

where $n\in\mathbb{Z}^+$, p is prime.

So $p=(5m+3)^2-n^2$

$=(5m+3+n)(5m+3-n)$ ✓

If p is prime, $(5m+3-n)=1$ (the smaller factor)

so $5m+2=n$ ✓

and $p=(5m+3+n)=(5m+3+5m+2)$

$=5(2m+1)$ ✓

which is not prime.

Hence a contradiction.

$\therefore \nexists\, m\in\mathbb{Z}^+ : (5m+3)^2=n^2+p$.

(14) $l: \underset{\sim}{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}-2\\2\\-1\end{pmatrix}$, $R(1-2\lambda, 2+2\lambda, 3-\lambda)$

$Z(0,0,\mu)$

(i) $\overrightarrow{RZ} \cdot z\text{-axis} = \begin{pmatrix}-1+2\lambda\\-2-2\lambda\\\mu-3+\lambda\end{pmatrix} \cdot \begin{pmatrix}0\\0\\\mu\end{pmatrix} = 0$

$\mu \neq 0$, $\lambda \neq 0$

ie $\mu(\mu-3+\lambda) = 0 \Rightarrow \mu+\lambda = 3$ (1)

(ii) $\overrightarrow{RZ} \cdot \begin{pmatrix}-2\\2\\-1\end{pmatrix} = 2-4\lambda-4-4\lambda-\mu+3-\lambda = 0$

ie $-\mu-9\lambda = -1$ (2)

(1) + (2) $\Rightarrow -8\lambda = 2 \Rightarrow \lambda = -\frac{1}{4}, \mu = \frac{13}{4}$

(iii) R is $\left(1-2(-\frac{1}{4}), 2+2(-\frac{1}{4}), 3-(-\frac{1}{4})\right)$

ie $\left(\frac{3}{2}, \frac{3}{2}, \frac{13}{4}\right)$, $Z\left(0,0,\frac{13}{4}\right)$

Shortest Distance $= \sqrt{(\frac{3}{2})^2 + (\frac{3}{2})^2 + 0^2}$

$= \frac{3}{\sqrt{2}}$ or $\frac{3\sqrt{2}}{2}$ units.

(b) (i) RTP $a^3 - b^3 = (a-b)(a^2+ab+b^2)$

RHS $= a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3$

$= a^3 - b^3$

$=$ LHS

(ii) RTP $(a^7+b^7)(a^2+b^2) \geq (a^5+b^5)(a^4+b^4)$

LHS − RHS $= a^9 + a^7b^2 + a^2b^7 + b^9 - (a^9 + a^5b^4 + a^4b^5 + b^9)$

$= a^7b^2 - a^4b^5 - a^5b^4 + a^2b^7$

$= a^4b^2(a^3-b^3) - a^2b^4(a^3-b^3)$

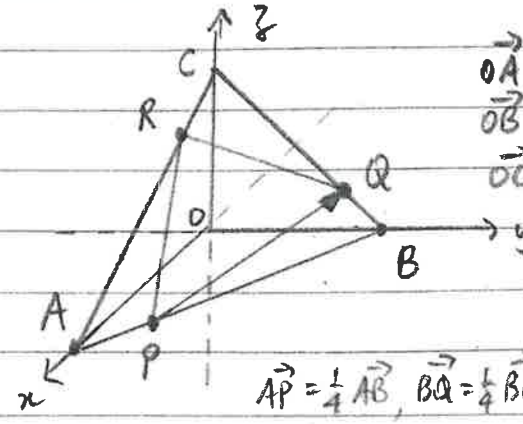
$= (a^3-b^3)a^2b^2(a^2-b^2)$

$= (a-b)(a^2+ab+b^2)a^2b^2(a+b)(a-b)$

$= a^2b^2(a-b)^2(a+b)(a^2+ab+b^2)$

≥ 0 for $a, b \in Z^+$ So proven

(c)



$\overrightarrow{OA} = 4\underset{\sim}{a}$

$\overrightarrow{OB} = 4\underset{\sim}{b}$

$\overrightarrow{OC} = 4\underset{\sim}{c}$

$\overrightarrow{AP} = \frac{1}{4}\overrightarrow{AB}$, $\overrightarrow{BQ} = \frac{1}{4}\overrightarrow{BC}$, $\overrightarrow{CR} = \frac{1}{4}\overrightarrow{CA}$

(i) $\overrightarrow{AP} = \frac{1}{4}(4\underset{\sim}{b} - 4\underset{\sim}{a}) = \underset{\sim}{b} - \underset{\sim}{a}$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = 4\underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{a} = 3\underset{\sim}{a} + \underset{\sim}{b}$

$\overrightarrow{BQ} = \frac{1}{4}(4\underset{\sim}{c} - 4\underset{\sim}{b}) = \underset{\sim}{c} - \underset{\sim}{b}$

$\overrightarrow{OQ} = \overrightarrow{OB} + \overrightarrow{BQ} = 4\underset{\sim}{b} + \underset{\sim}{c} - \underset{\sim}{b} = 3\underset{\sim}{b} + \underset{\sim}{c}$ show

$\therefore \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 3\underset{\sim}{b} + \underset{\sim}{c} - (3\underset{\sim}{a} + \underset{\sim}{b})$

$= -3\underset{\sim}{a} + 2\underset{\sim}{b} + \underset{\sim}{c}$

(ii) Given $\overrightarrow{QR} = -3\underset{\sim}{b} + 2\underset{\sim}{c} + \underset{\sim}{a}$

$\overrightarrow{PQ} \cdot \overrightarrow{QR} = (-3\underset{\sim}{a} + 2\underset{\sim}{b} + \underset{\sim}{c}) \cdot (-3\underset{\sim}{b} + 2\underset{\sim}{c} + \underset{\sim}{a})$

$= 9\underset{\sim}{a}\cdot\underset{\sim}{b} - 6\underset{\sim}{a}\cdot\underset{\sim}{c} - 3|\underset{\sim}{a}|^2 - 6|\underset{\sim}{b}|^2 + 4\underset{\sim}{b}\cdot\underset{\sim}{c}$
$+ 2\underset{\sim}{a}\cdot\underset{\sim}{b} - 3\underset{\sim}{b}\cdot\underset{\sim}{c} + 2|\underset{\sim}{c}|^2 + \underset{\sim}{a}\cdot\underset{\sim}{c}$

$= -3|\underset{\sim}{a}|^2 - 6|\underset{\sim}{b}|^2 + 2|\underset{\sim}{c}|^2$ since

$\underset{\sim}{a}\cdot\underset{\sim}{b} = \underset{\sim}{b}\cdot\underset{\sim}{c} = \underset{\sim}{a}\cdot\underset{\sim}{c} = 0$

(axis perpendicular).

(iii) If $\angle PQR = 90°$, $\overrightarrow{PQ} \cdot \overrightarrow{QR} = 0$, $|\underset{\sim}{a}| = 2, |\underset{\sim}{b}| = 1$

$\Rightarrow 0 = -3(2)^2 - 6(1)^2 + 2|\underset{\sim}{c}|^2$

$\Rightarrow 18 = 2|\underset{\sim}{c}|^2 \Rightarrow |\underset{\sim}{c}|^2 = 9 \Rightarrow |\underset{\sim}{c}| = 3$

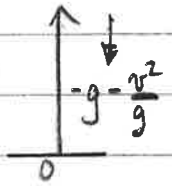
So $\underset{\sim}{a} = \begin{pmatrix}2\\0\\0\end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\underset{\sim}{c} = \begin{pmatrix}0\\0\\3\end{pmatrix}$

$\overrightarrow{PQ} = \begin{pmatrix}-6\\2\\3\end{pmatrix}$, $|\overrightarrow{PQ}|^2 = 36 + 4 + 9 = 49$

$\overrightarrow{QR} = \begin{pmatrix}2\\-3\\6\end{pmatrix}$, $|\overrightarrow{QR}|^2 = 4 + 9 + 36 = 49$

$\therefore |\overrightarrow{PQ}| = |\overrightarrow{QR}| = 7$, $\triangle PQR$ isosceles.

15 (a)



$t=0,\ x=0,\ v=\sqrt{3}g$

$\ddot{x} = -g - \frac{v^2}{g}$

(i) $\frac{dv}{dt} = -\frac{g^2+v^2}{g}$

$t = -g\int \frac{1}{g^2+v^2}\,dv$

$= -\tan^{-1}\left(\frac{v}{g}\right) + c_1$ ✓

$t=0,\ v=\sqrt{3}g \Rightarrow c_1 = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

$t = \frac{\pi}{3} - \tan^{-1}\left(\frac{v}{g}\right)$

$\tan^{-1}\left(\frac{v}{g}\right) = \frac{\pi}{3} - t$

$v = g\tan\left(\frac{\pi}{3}-t\right)$ ✓ (1)

(ii) $x = \int g\tan\left(\frac{\pi}{3}-t\right)dt$

$= g\int \frac{\sin\left(\frac{\pi}{3}-t\right)}{\cos\left(\frac{\pi}{3}-t\right)}dt$

$= g\ln\left|\cos\left(\frac{\pi}{3}-t\right)\right| + c_2$ ✓

$t=0,\ x=0 \Rightarrow c_2 = -g\ln\left|\cos\frac{\pi}{3}\right|$

$= -g\ln\left(\frac{1}{2}\right)$

$= g\ln 2$

$\therefore\ x = g\ln\left[2\left|\cos\left(\frac{\pi}{3}-t\right)\right|\right]$ (2) ✓

(iii) $v=0$ in (1) $\Rightarrow t=\frac{\pi}{3}$ then in (2)

$x = g\ln 2\cos 0$

$= g\ln 2$ Max height. ✓

(iv) $v\cdot\frac{dv}{dx} = -\frac{g^2+v^2}{g}$

$x = \int \frac{-gv}{g^2+v^2}\,dv$

$= -\frac{g}{2}\ln\left|g^2+v^2\right| + c_3$ ✓

$x=0,\ v=\sqrt{3}g \Rightarrow c_3 = \frac{g}{2}\ln 4g^2$

So $x = \frac{g}{2}\ln\left(\frac{4g^2}{g^2+v^2}\right)$ ✓

and $v=0 \Rightarrow x = \frac{g}{2}\ln 4 = \frac{g}{2}\ln 2^2 = g\ln 2$

(b) $1, -1, -5, -7, 1, 23, \ldots$

$T_{n+2} = 2T_{n+1} - 3T_n \quad , T_1 = 1,\ T_2 = -1$

(i) $T_7 = 2T_6 - 3T_5$

$= 2\times 23 - 3\times 1$

$= 43$ ✓

(ii) $T_n = \frac{1}{2}\left((1+i\sqrt{2})^n + (1-i\sqrt{2})^n\right)$

$T_1 = \frac{1}{2}\left((1+i\sqrt{2})^1 + (1-i\sqrt{2})^1\right) = 1$ ✓

$T_2 = \frac{1}{2}\left((1+i\sqrt{2})^2 + (1-i\sqrt{2})^2\right)$

$= \frac{1}{2}\left(1+2\sqrt{2}i-2 + 1-2\sqrt{2}i-2\right)$ Show

$= \frac{1}{2}(-2) = -1$ ✓

(iii) In (ii), we have shown the result is true when $n=1$ and $n=2$.

Assume it is true for $n=k$ and $n=k+1$, and try to show it is true for $n=k+2$

i.e. Assume $T_k = \frac{1}{2}\left((1+i\sqrt{2})^k + (1-i\sqrt{2})^k\right)$

and $T_{k+1} = \frac{1}{2}\left((1+i\sqrt{2})^{k+1} + (1-i\sqrt{2})^{k+1}\right)$

RTP $T_{k+2} = \frac{1}{2}\left((1+i\sqrt{2})^{k+2} + (1-i\sqrt{2})^{k+2}\right)$

Now $T_{k+2} = 2T_{k+1} - 3T_k$

$= 2\times\frac{1}{2}\left((1+i\sqrt{2})^{k+1} + (1-i\sqrt{2})^{k+1}\right)$

$-3\times\frac{1}{2}\left((1+i\sqrt{2})^k + (1-i\sqrt{2})^k\right)$

(15) (b) (iii) (continued)

$$T_{k+2} = \frac{1}{2}\left[2(1+\sqrt{2}i)(1+\sqrt{2}i)^k - 3(1+\sqrt{2}i)^k + 2(1-\sqrt{2}i)(1-\sqrt{2}i)^k - 3(1-\sqrt{2}i)^k\right]$$

$$= \frac{1}{2}\left[(1+\sqrt{2}i)^k(2+2\sqrt{2}i-3) + (1-\sqrt{2}i)^k(2-2\sqrt{2}i-3)\right]$$

$$= \frac{1}{2}\left[(1+\sqrt{2}i)^k(1+2\sqrt{2}i-2) + (1-\sqrt{2}i)^k(1-2\sqrt{2}i-2)\right]$$

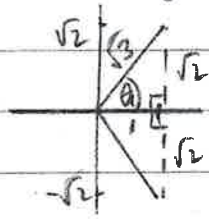
$$= \frac{1}{2}\left[(1+\sqrt{2}i)^k(1+\sqrt{2}i)^2 + (1-\sqrt{2}i)^k(1-\sqrt{2}i)^2\right]$$

$$= \frac{1}{2}\left((1+\sqrt{2}i)^{k+2} + (1-\sqrt{2}i)^{k+2}\right)$$

as required.

$\therefore$ If the result is true for $n=k, k+1$, it is also true for $n=k+2$, Since true for $n=1,2$, by the process of Mathematical Induction, it is true for all $n \in Z^+$

(iv) $T_n = \frac{1}{2}\left((1+\sqrt{2}i)^n + (1-\sqrt{2}i)^n\right)$



$\tan\theta = \sqrt{2}$, $\theta = \tan^{-1}\sqrt{2}$

$\therefore (1+\sqrt{2}i) = \sqrt{3}\,\text{cis}\,\theta$

$(1-\sqrt{2}i) = \sqrt{3}\,\text{cis}\,(-\theta)$

$$\therefore T_n = \frac{1}{2}\left((\sqrt{3}\,\text{cis}\,\theta)^n + (\sqrt{3}\,\text{cis}(-\theta))^n\right)$$

De Moivre

$$= \frac{1}{2}(\sqrt{3})^n\left(\cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)\right)$$

$$= \frac{1}{2}(\sqrt{3})^n\left(\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta\right)$$

$$= \frac{1}{2}(\sqrt{3})^n(2\cos n\theta)$$

$$= (\sqrt{3})^n \cos n\left(\tan^{-1}\sqrt{2}\right)$$

16 (a) $I=\int_0^1 \sqrt{x-x^2}\,dx$

(i)

Let $x-\frac{1}{2}=\frac{1}{2}\sin\theta$

$dx=\frac{1}{2}\cos\theta\,d\theta$

$x=0,\ \theta=-\frac{\pi}{2}$; $x=1,\ \theta=\frac{\pi}{2}$

$I=\int_0^1\sqrt{\frac{1}{4}-(x-\frac{1}{2})^2}\,dx$

$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\frac{1}{4}-\frac{1}{4}\sin^2\theta}\ \frac{1}{2}\cos\theta\,d\theta$ ✓

$=\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta$

$=\frac{1}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\cos2\theta)\,d\theta$ ✓

No marks for a substitution that led nowhere

$=\frac{1}{8}\left[\theta+\frac{1}{2}\sin2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$=\frac{1}{8}\left(\frac{\pi}{2}+0-(-\frac{\pi}{2}+0)\right)$

$=\frac{\pi}{8}$ ✓

OR

$I=\int_0^1\sqrt{x-x^2}\,dx$

Let $x=\sin^2\theta$

$dx=2\sin\theta\cos\theta\,d\theta$

$x=0,\ \theta=0$; $x=1,\ \theta=\frac{\pi}{2}$

$I=\int_0^{\frac{\pi}{2}}\sqrt{\sin^2\theta(1-\sin^2\theta)}\ 2\sin\theta\cos\theta\,d\theta$ ✓

$=\int_0^{\frac{\pi}{2}}2(\sin\theta\cos\theta)^2\,d\theta$

$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^2 2\theta\,d\theta$

$=\frac{1}{4}\int_0^{\frac{\pi}{2}}(1-\cos4\theta)\,d\theta$ ✓

$=\frac{1}{4}\left[\theta-\frac{1}{4}\sin4\theta\right]_0^{\frac{\pi}{2}}$

$=\frac{1}{4}\left[\frac{\pi}{2}-0-(0-0)\right]$

$=\frac{\pi}{8}$ ✓

(ii) $I_n=\int_0^1 x^{n+\frac{1}{2}}\sqrt{1-x}\,dx$, integral $n\geq 0$.

Let $u=x^{n+\frac{1}{2}}$, $v'=(1-x)^{\frac{1}{2}}$

$u'=(n+\frac{1}{2})x^{n-\frac{1}{2}}$, $v=-\frac{2}{3}(1-x)^{\frac{3}{2}}$

$I_n=\left[-\frac{2}{3}x^{n+\frac{1}{2}}(1-x)^{\frac{3}{2}}\right]_0^1+\frac{2}{3}(n+\frac{1}{2})\int_0^1(1-x)^{\frac{3}{2}}x^{n-\frac{1}{2}}\,dx$ ✓

$=[0-0]+\frac{2n+1}{3}\int_0^1\sqrt{1-x}\,(1-x)x^{n-\frac{1}{2}}\,dx$

$=\frac{2n+1}{3}\left(\int_0^1\sqrt{1-x}\,(x^{n-\frac{1}{2}}-x^{n+\frac{1}{2}})\,dx\right)$

$3I_n=(2n+1)(I_{n-1}-I_n)$

$(2n+4)I_n=(2n+1)I_{n-1}$

$\therefore I_n=\frac{2n+1}{2n+4}I_{n-1}$ ✓

(iii) $\int_0^1 x^n\sqrt{x-x^2}\,dx=\int_0^1 x^{n+\frac{1}{2}}\sqrt{1-x}\,dx=I_n$

$I_0=\frac{\pi}{8}$ from (i)

$=\frac{(2n+1)}{(2n+4)}\times\frac{2(n-1)+1}{2(n-1)+4}\times I_{n-2}$

$=\frac{(2n+1)(2n-1)(2n-3)\cdots5\times3}{(2n+4)(2n+2)(2n)\cdots8\times6}\times\frac{8I_0}{4\times2}$ ✓

need to be shown

$=\frac{(2n+1)(2n-1)(2n-3)\cdots5\times3}{2^{n+2}(n+2)!}\times\frac{(2n)(2n-2)\cdots4\times2}{2^n\,n!}\times\pi$ ✓

$=\frac{(2n+1)(2n)(2n-1)(2n-2)\cdots5\times4\times3\times2\times\pi}{2^{2n+2}(n+2)!\,n!}$

$=\frac{(2n+1)!\,\pi}{2^{2n+2}(n+2)!\,n!}$ ✓

(Continued)

16 (a) (iii) OR

Using Mathematical Induction (outline only here)

RTP $I_n = \dfrac{(2n+1)!\,\pi}{2^{2n+2}(n+2)!\,n!}$

Base Case: $I_0 = \dfrac{1!\,\pi}{2^2\,2!\,0!} = \dfrac{\pi}{8}$

Assume $I_k = \dfrac{(2k+1)!\,\pi}{2^{2k+2}(k+2)!\,k!}$

Try to show $I_{k+1} = \dfrac{(2k+3)!\,\pi}{2^{2k+4}(k+3)!(k+1)!}$

Relatively straightforward using the recurrence relation $I_{k+1} = \dfrac{(2(k+1)+1)}{(2(k+1)+4)} I_k$

(b) Let $c = \cos\theta$, $s = \sin\theta$

(i) By de Moivre, $(c+is)^{2k} = \cos 2k\theta + i\sin 2k\theta$

Also

$(c+is)^{2k} = c^{2k} + \binom{2k}{1}c^{2k-1}is - \binom{2k}{2}c^{2k-2}s^2 - \binom{2k}{3}c^{2k-3}is^3 + \cdots$

Equating Imaginary parts.

$\sin 2k\theta = \binom{2k}{1}c^{2k-1}s - \binom{2k}{3}c^{2k-3}s^3 + \binom{2k}{5}c^{2k-5}s^5 - \cdots$

$\therefore \dfrac{\sin 2k\theta}{\sin\theta\cos\theta} = \binom{2k}{1}c^{2k-2} - \binom{2k}{3}c^{2k-4}s^2 + \binom{2k}{5}c^{2k-6}s^4 - \cdots$

$= \binom{2k}{1}(1-s^2)^{k-1} - \binom{2k}{3}(1-s^2)^{k-2}s^2 + \binom{2k}{5}(1-s^2)^{k-3}s^4 \cdots$

Hence $\dfrac{\sin 2k\theta}{\sin\theta\cos\theta}$ is a polynomial in $\sin^2\theta$.

(ii) For $k=4$, $\dfrac{\sin 8\theta}{\sin\theta\cos\theta} = \binom{8}{1}(1-s^2)^3 - \binom{8}{3}(1-s^2)^2s^2 + \binom{8}{5}(1-s^2)(s^2)^2 - \binom{8}{7}(s^2)^3$

$= 8(1-3s^2+3s^4-s^6) - 56(s^2-2s^4+s^6) + 56(s^4-s^6) - 8s^6$

$= 8 - 24s^2 + 24s^4 - 8s^6 - 56s^2 + 112s^4 - 56s^6 + 56s^4 - 56s^6 - 8s^6$

$= -2(64s^6 - 96s^4 + 40s^2 - 4)$

For $z^6 - 6z^4 + 10z^2 - 4 = 0$, Let $z = 2\sin\theta = 2s$

So $64s^6 - 96s^4 + 40s^2 - 4 = 0$

$\Rightarrow \sin 8\theta = 0$, but $\sin\theta \neq 0$, $\cos\theta \neq 0$

$\therefore 8\theta = k\pi$, $k \neq 4n$, $n \in \mathbb{Z}$

$\theta = \dfrac{k\pi}{8}$

Let $k = -3, -2, -1, 1, 2, 3$

$\theta = -\frac{3\pi}{8}, -\frac{\pi}{4}, -\frac{\pi}{8}, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}$

So $z = 2\sin(-\frac{3\pi}{8}), 2\sin(-\frac{\pi}{4}), 2\sin(-\frac{\pi}{8}), 2\sin(\frac{\pi}{8}), 2\sin(\frac{\pi}{4}), 2\sin(\frac{3\pi}{8})$

So $z = \pm 2\sin\frac{\pi}{8}, \pm\sqrt{2}, \pm 2\sin\frac{3\pi}{8}$

SYDNEY GRAMMAR SCHOOL

NAME

MATHS MASTER

CANDIDATE NUMBER

2024 Trial HSC Examination

Form VI Mathematics Extension 2

Tuesday 13th August 2024
8:40am

General Instructions

- Reading time — 10 minutes
- Working time — 3 hours
- Attempt all questions.
- Write using black pen.
- Calculators approved by NESA may be used.
- A loose reference sheet is provided separate to this paper.

Total Marks: 100

Section I (10 marks) Questions 1 – 10
- This section is multiple-choice. Each question is worth 1 mark.
- Record your answers on the provided answer sheet.

Section II (90 marks) Questions 11 – 16
- Relevant mathematical reasoning and calculations are required.
- Start each question in a new booklet.

Collection

- Your name and master should only be written on this page.
- Write your candidate number on this page, on each booklet and on the multiple choice sheet.
- If you use multiple booklets for a question, place them inside the first booklet for the question.
- Arrange your solutions in order.

Checklist
- Reference sheet
- Multiple-choice answer sheet
- 6 booklets per boy
- Candidature: 77 pupils

Writer: PC

Section I

Questions in this section are multiple-choice.
Record the single best answer for each question on the provided answer sheet.

---

1. In the Argand diagram, the complex number z lies in the second quadrant. In which quadrant does the complex number $i\,\overline{z}$ lie?

   (A) first

   (B) second

   (C) third

   (D) fourth

2. Given $\underset{\sim}{u} = \underset{\sim}{i} + 2\underset{\sim}{j}$, $\underset{\sim}{v} = \underset{\sim}{j} + 3\underset{\sim}{k}$, what is the projection of $\underset{\sim}{u}$ onto $\underset{\sim}{v}$?

   (A) $\dfrac{1}{5}(\underset{\sim}{i} + 2\underset{\sim}{j})$

   (B) $\dfrac{1}{5}(\underset{\sim}{j} + 3\underset{\sim}{k})$

   (C) $\dfrac{7}{10}(\underset{\sim}{i} + 2\underset{\sim}{j})$

   (D) $\dfrac{7}{10}(\underset{\sim}{j} + 3\underset{\sim}{k})$

3. A particle is moving in Simple Harmonic Motion with amplitude 3 metres. Its speed is 4 metres per second when the particle is 1 metre from the centre of motion. What is the period of the motion?

   (A) $\dfrac{\pi}{2}$

   (B) $\dfrac{\pi}{\sqrt{2}}$

   (C) $\sqrt{2}\pi$

   (D) 2π

4. Consider the identity: $\dfrac{8}{(x+1)(x-1)^2} \equiv \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$.

   Which of the following are the correct values of A, B and C?

   (A) $A = 2,\ B = -2,\ C = 4$

   (B) $A = 2,\ B = -2,\ C = -4$

   (C) $A = -2,\ B = 2,\ C = 4$

   (D) $A = -2,\ B = 2,\ C = -4$

5. Let $\overrightarrow{OP} = \frac{1}{2}(\sqrt{2}\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k})$, and α, β and γ be the angles that $\overrightarrow{OP}$ makes with the positive x, y and z-axes respectively. What is the value of $\alpha + \beta + \gamma$?

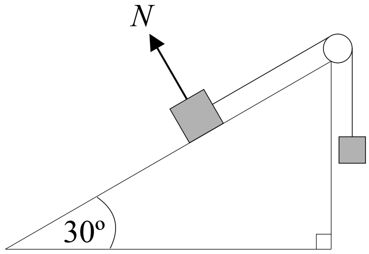
   (A) $45°$

   (B) $165°$

   (C) $180°$

   (D) $225°$

6.



Two masses are attached to a light inextensible string which is looped over a smooth pulley as shown in the diagram. The larger mass is on a smooth incline of $30°$ to the horizontal, and the smaller mass (M kg) hangs freely. The masses are stationary and at equilibrium, and the magnitude of the acceleration due to gravity is $g\,\text{m/s}^2$. What is the magnitude, in Newtons, of the normal reaction force N, on the larger mass?

   (A) $N = \frac{\sqrt{3}}{2}Mg$

   (B) $N = Mg$

   (C) $N = \sqrt{3}Mg$

   (D) $N = 2Mg$

7. Given that $|z| = 2$, what is the greatest possible value of $\operatorname{Arg}(z + 4i)$?

   (A) $\dfrac{\pi}{6}$

   (B) $\dfrac{\pi}{3}$

   (C) $\dfrac{2\pi}{3}$

   (D) $\dfrac{5\pi}{6}$

8. A single die is rolled and the uppermost face noted. Let p represent the statement "the uppermost face is divisible by 3" and let q represent the statement "the uppermost face is divisible by 6". Considering the implication $p \Rightarrow q$, which of the following is correct?

   (A) The negation is true and the converse is true.

   (B) The negation is true and the converse is false.

   (C) The negation is false and the converse is true.

   (D) The negation is false and the converse is false.

9. If w is the complex root of $z^5 = 1$ with smallest positive argument, which of the following is false?

   (A) $\text{Re}(w + w^3) < 0$

   (B) $\text{Im}(w + w^3) > 0$

   (C) $\text{Re}(w + w^4) > 0$

   (D) $\text{Im}(w + w^4) < 0$

10. Given that x and y are real numbers, which of the following is a true statement?

    (A) $\forall y$, $\exists x$ such that $x^2 - y^2 = x$

    (B) $\forall y$, $\exists x$ such that $x^2 - y^2 = y$

    (C) $\forall y$, $\exists x$ such that $x^2 + y^2 = x$

    (D) $\forall y$, $\exists x$ such that $x^2 + y^2 = y$

End of Section I

The paper continues in the next section

Section II

This section consists of long-answer questions.
Marks may be awarded for reasoning and calculations.
Marks may be lost for poor setting out or poor logic.
Start each question in a new booklet.

QUESTION ELEVEN (15 marks) Start a new answer booklet. **Marks**

(a) Consider two complex numbers $z = a + 2i$ and $w = 1 - ai$, where a is real.

(i) Find zw in the form $x + iy$. 1

(ii) Find $z - aw$ in modulus-argument form. 1

(iii) Show that $(\overline{w})^2 + 2w$ is real. 1

(b) Find the indefinite integrals:

(i) $\displaystyle\int \frac{e^x}{1 + e^{2x}}\,dx$ 1

(ii) $\displaystyle\int \sin^4 x \sin 2x\,dx$ 1

(c) Sketch the region in the complex plane where the inequalities $\text{Re}(z) < 1$, $\text{Re}(z) < \text{Im}(z)$, and $-\dfrac{\pi}{2} < \text{Arg}\,(z + 1) < \dfrac{\pi}{4}$ all hold simultaneously. 3

(d) A particle moving along the x-axis has acceleration a, velocity v and displacement x at time t. Initially, $x = 0$ and $v = 2$.

(i) If $v = x^2 + 1$, find a when $x = 3$. 2

(ii) If $a = x^2 + 1$, find v when $x = 3$. 3

(e) Consider the complex numbers u, v and z, such that $u = 2i$, $|v| = 3$ and $z = uv$. Find the exact value of $|z - v|$. 2

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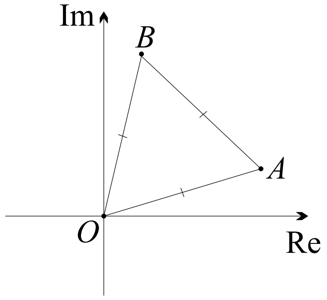
QUESTION TWELVE (15 marks) Start a new answer booklet. **Marks**

(a)



Let O, A and B be points in the complex plane representing the numbers 0, $6+2i$ and z. If $\triangle OAB$ is equilateral, with vertices in anti-clockwise order, determine the exact value of z in the form $x+iy$. **2**

(b) (i) Use the substitution $t = \tan\frac{x}{2}$ to show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x+\sin x} = \ln 2\,.$ **2**

(ii) Hence use the substitution $x = \frac{\pi}{2} - u$, to find the exact value of **3**

$$\int_0^{\frac{\pi}{2}} \frac{x\,dx}{1+\cos x+\sin x}\,.$$

(c) Given $z^4 - 2z^3 + 9z^2 - 6z + 18 = 0$ has $1 + i\sqrt{5}$ as a root, find all the roots. **3**

(d) Consider the line l with equation $\underset{\sim}{r} = \begin{bmatrix} 7 \\ 4 \\ 13 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 7 \\ 10 \end{bmatrix}$ and the sphere S with equation $\left| \underset{\sim}{r} - \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \right| = 3$. Show that l touches S and find the point of contact Q. **3**

(e) Prove, using contraposition, that $\forall\, x, y \in \mathbf{Z}$, if $x^2(y+3)$ is even, then x is even or y is odd. **2**

The paper continues on the next page

QUESTION THIRTEEN (15 marks) Start a new answer booklet. **Marks**

(a) Use integration by parts twice to find $\displaystyle\int x^3(\log x)^2\,dx$. **4**

(b) Show that $(\cos\theta + i\sin\theta)^n(\sin\theta + i\cos\theta)^n = e^{\frac{ni\pi}{2}}$. **2**

(c) (i) Show that $\dfrac{2}{(x+1)(x^2+1)} \equiv \dfrac{1}{x+1} - \dfrac{x-1}{x^2+1}$. **1**

(ii) Let $I_n = \displaystyle\int_0^1 \frac{2x^n}{(x+1)(x^2+1)}\,dx$.

(α) Show that $I_0 = \dfrac{1}{2}\log 2 + \dfrac{\pi}{4}$. **2**

(β) By considering $I_0 + I_2$, or otherwise, find the exact value of I_2. **2**

(d) (i) Give an example of positive integers m, n and p, where p is prime, such that $(2m+3)^2 = n^2 + p$. **1**

(ii) Use proof by contradiction to show that if p is prime and n is a positive integer, then no positive integer m exists such that $(5m+3)^2 = n^2 + p$. **3**

The paper continues on the next page

QUESTION FOURTEEN (15 marks) Start a new answer booklet. **Marks**

(a) Consider a typical point $R(1-2\lambda, 2+2\lambda, 3-\lambda)$ on the line l: $\underset{\sim}{r} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$, and a typical point $Z(0,0,\mu)$ on the z-axis, where λ and μ are non-zero parameters.

(i) Show that $\overrightarrow{RZ}$ is perpendicular to the z-axis when $\mu + \lambda = 3$. **1**

(ii) Find the values of μ and λ such that $\overrightarrow{RZ}$ is perpendicular to both l and the z-axis. **2**

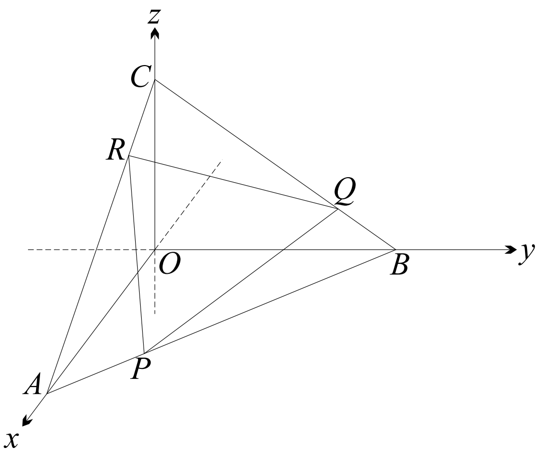
(iii) Hence find the shortest distance between l and the z-axis. **2**

(b) (i) Show that $a^3 - b^3 = (a-b)(a^2+ab+b^2)$. **1**

(ii) Hence, or otherwise, show that for positive integers a and b, **3**

$$(a^7+b^7)(a^2+b^2) \geq (a^5+b^5)(a^4+b^4).$$

(c)



In the diagram, A, B and C lie on the positive x, y and z-axes respectively, and let $\overrightarrow{OA} = 4\underset{\sim}{a}$, $\overrightarrow{OB} = 4\underset{\sim}{b}$, $\overrightarrow{OC} = 4\underset{\sim}{c}$, $\overrightarrow{AP} = \frac{1}{4}\overrightarrow{AB}$, $\overrightarrow{BQ} = \frac{1}{4}\overrightarrow{BC}$ and $\overrightarrow{CR} = \frac{1}{4}\overrightarrow{CA}$.

(i) Show that $\overrightarrow{PQ} = -3\underset{\sim}{a} + 2\underset{\sim}{b} + \underset{\sim}{c}$. **2**

(ii) It can also be shown that $\overrightarrow{QR} = -3\underset{\sim}{b} + 2\underset{\sim}{c} + \underset{\sim}{a}$ (**do not** prove this). **2**
Show that $\overrightarrow{PQ} \cdot \overrightarrow{QR} = -3|\underset{\sim}{a}|^2 - 6|\underset{\sim}{b}|^2 + 2|\underset{\sim}{c}|^2$.

(iii) Given $|\underset{\sim}{a}| = 2$ and $|\underset{\sim}{b}| = 1$, show that if $\triangle PQR$ is right angled at Q, then it is also isosceles. **2**

The paper continues on the next page

QUESTION FIFTEEN (15 marks) Start a new answer booklet. **Marks**

(a) A projectile of unit mass is launched vertically upwards from the origin with an initial velocity of $\sqrt{3}\,g$ m/s, where g m/s^2 is the acceleration due to gravity. The projectile experiences a resistive force of magnitude $\dfrac{v^2}{g}$ Newtons, where v m/s is the velocity of the particle after t seconds. The acceleration of the projectile is given by

$$\ddot{x} = -g - \frac{v^2}{g}.$$

(i) Show that $v = g\tan\left(\frac{\pi}{3} - t\right)$. **2**

(ii) Find an expression for the displacement x metres in terms of g and t. **2**

(iii) Show that the maximum height achieved by the particle is $g\ln 2$ metres. **1**

(iv) Derive an expression for x in terms of v^2 and show that this equation confirms the maximum height found in part (iii). **2**

(b) Consider the sequence of numbers: $1,\ -1,\ -5,\ -7,\ 1,\ 23,\ \ldots$

These numbers can be generated using the recurrence relation

$$T_{n+2} = 2\,T_{n+1} - 3\,T_n, \text{ for } n \geq 1 \text{ with } T_1 = 1 \text{ and } T_2 = -1.$$

(i) Use the recurrence relation to find T_7. **1**

(ii) Show that the formula $T_n = \dfrac{\left(1 + i\sqrt{2}\,\right)^n + \left(1 - i\sqrt{2}\,\right)^n}{2}$ generates T_1 and T_2. **2**

(iii) Use Mathematical Induction to prove the formula for T_n in part (ii) works for all positive integers n. **3**

(iv) Show that the formula for T_n in part (ii) is equivalent to **2**

$$T_n = \left(\sqrt{3}\right)^n \cos\left(n\tan^{-1}\sqrt{2}\right).$$

The paper continues on the next page

QUESTION SIXTEEN (15 marks) Start a new answer booklet. **Marks**

(a) (i) Using a suitable trigonometric substitution, or otherwise, show that **3**

$$\int_0^1 \sqrt{x - x^2}\, dx = \frac{\pi}{8}.$$

(ii) Given the integral $I_n = \displaystyle\int_0^1 x^{n+\frac{1}{2}} \sqrt{1-x}\, dx$, for integers $n \geq 0$, show that **2**

$$I_n = \frac{2n+1}{2n+4}\, I_{n-1}.$$

(iii) Show that for integers $n \geq 0$: **3**

$$\int_0^1 x^n \sqrt{x - x^2}\, dx = \frac{(2n+1)!\,\pi}{2^{2n+2}\,(n+2)!\,n!}\,.$$

(b) (i) Using de Moivre's Theorem, or otherwise, show that $\dfrac{\sin 2k\theta}{\sin\theta\cos\theta}$, where k is a positive integer, can always be expressed as a polynomial in $\sin^2\theta$. **3**

(ii) Obtain the polynomial in $\sin^2\theta$ corresponding to $\dfrac{\sin 8\theta}{\sin\theta\cos\theta}$, and hence solve the equation: $z^6 - 6z^4 + 10z^2 - 4 = 0$. **4**

END OF PAPER